The **Complexity** of **Limited Belief Reasoning**

The Quantifier-Free Case

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Motivating Limited Belief Reasoning

Problem: Does KB logically entail α?

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Classical logic:

- Unrealistic: omniscience!
- Hard: co-NP-hard (propositional) / undecidable (first-order)

Limited belief:

- Reasoning budget $\rightarrow$ No omniscience
- More realistic
- More tractable

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Limited belief:

- Reasoning budget $\rightarrow$ No omniscience
- More realistic (Yes! [IJCAI 2017])
- More tractable (Actually... [this paper])

What is Limited Belief?

- Functions and equality
- And, or, not, quantifiers
- Modalities for belief and action, possibly nested

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uninterpreted constant
unique name

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Limited Belief Reasoning: does KB (in CNF) entail $\mathbf{B}_k\,\alpha$?

- Belief level 0: clause subsumption
- Belief level $k+1$: case split + belief level k

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- $\mathrm{KB} \models \mathbf{B}_0\,\alpha$ iff all subclauses of α subsumed after unit propagation
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Example:
$$\begin{aligned}\mathrm{KB} \; := \; & (\mathrm{father} = \mathrm{Frank} \vee \mathrm{father} = \mathrm{Fred}) \wedge \\ & (\mathrm{father} = \mathrm{Frank} \rightarrow \mathrm{mother} = \mathrm{Mia}) \wedge \\ & (\mathrm{father} = \mathrm{Fred} \rightarrow \mathrm{mother} = \mathrm{Molly})\end{aligned}$$

$$\mathrm{KB} \not\models \mathbf{B}_0\,(\mathrm{mother} = \mathrm{Mia} \vee \mathrm{mother} = \mathrm{Molly}) \quad ✗$$

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$$\mathrm{KB} \land \mathrm{father} = \mathrm{Frank} \models \mathbf{B}_0\,\mathrm{mother} = \mathrm{Mia} \qquad \checkmark$$

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$\mathrm{KB} \land \mathrm{father} = \mathrm{Frank} \models \mathbf{B}_0\,\mathrm{mother} = \mathrm{Mia}$ ✓
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$$\begin{array}{lll} \mathrm{KB} \land \mathrm{father} = \mathrm{Frank} & \models \mathbf{B}_0\,\mathrm{mother} = \mathrm{Mia} & \checkmark \\ \mathrm{KB} \land \mathrm{father} = \mathrm{Fred} & \models \mathbf{B}_0\,\mathrm{mother} = \mathrm{Molly} & \checkmark \\ \mathrm{KB} \land \mathrm{father} = n & \models \mathbf{B}_0\,\bot & \checkmark \end{array}$$

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$\mathrm{KB} \models \mathbf{B}_1 (\mathrm{mother} = \mathrm{Mia} \lor \mathrm{mother} = \mathrm{Molly})$ ✓

Classical Complexity

- $\mathrm{KB} \models \mathbf{B}_0\, \alpha$ iff all subclauses of α subsumed after unit propagation
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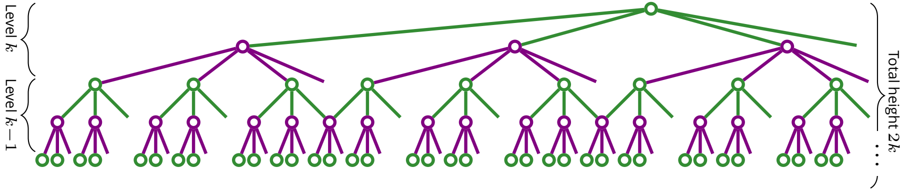

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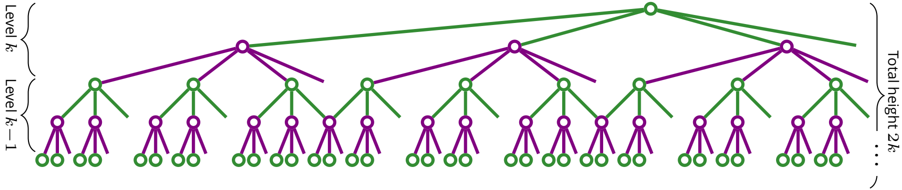

Theorem: KB $\models \mathbf{B}_k\,\alpha$ is in PTIME
if k or #Constants is constant

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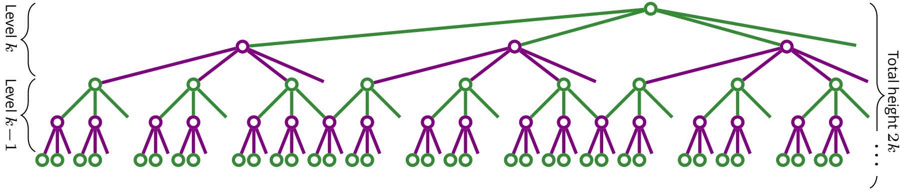

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Theorem: KB $\models \mathbf{B}_k\,\alpha$ is PSPACE

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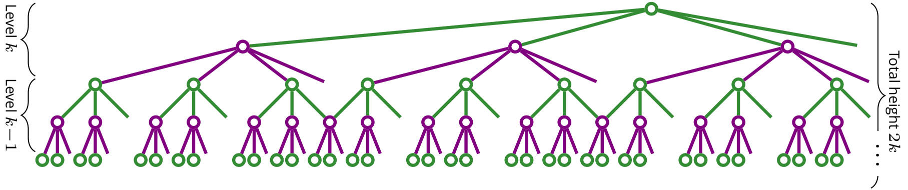


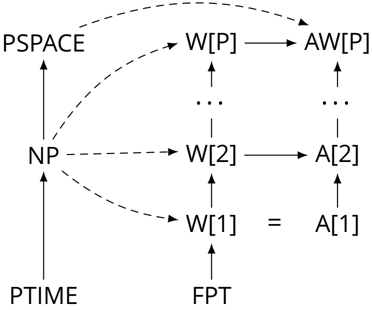
Theorem: $\mathrm{KB} \models \mathbf{B}_k\, \alpha$ is in PTIME
if k or #Constants is constant

Theorem: $\mathrm{KB} \models \mathbf{B}_k\, \alpha$ is PSPACE-complete

Order of splits matters:

$$c = n \vee d_1 = n \vee \ldots \qquad c \neq n \vee d_2 = n \vee \ldots$$
$$c = n \vee d_1 \neq n \vee \ldots \qquad c \neq n \vee d_2 \neq n \vee \ldots$$

Parameterized Complexity



FPT: k-vertex cover

$f(k) \cdot p(n)$ steps for computable f, polynomial p

W[1], A[1]: weighted 3CNF satisfiability

$f(k) \cdot p(n)$ steps with EXISTS steps at the end

(A)W[P]: (quantified) weighted circuit satisfiability

$f(k) \cdot p(n)$ steps + $g(k)$ EXISTS (+ FORALL) steps

Complexity Overview

$\mathrm{KB} \models \mathbf{B}_{k+1}\,\alpha$ iff for some constant c, for all names n, $\mathrm{KB} \wedge c = n \models \mathbf{B}_k\,\alpha$

#Constants	Belief level	#Names	
Input	Input	—	PSPACE-complete
—	Const	—	PTIME
Const	—	—	

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$\mathrm{KB} \models \mathbf{B}_{k+1}\,\alpha$ iff for some constant c, for all names n, $\mathrm{KB} \land c = n \models \mathbf{B}_k\,\alpha$

#Constants	Belief level	#Names	
Input	Input	—	PSPACE-complete
	Param	Input	AW[P]-complete
		Param Const	W[P]-complete
Param	Input Param	Input	co-W[P]-complete
	—	Param Const	FPT
— Const	Const —	— —	PTIME

Conclusions

Summary

- Only tractable for small k or #Constants.
- Otherwise, harder than classical logic!
- $\rightarrow$ Belief level limits a resource that must be used wisely.

Next Steps

- Change split rule to reduce complexity?
- $\in$ PSPACE $\rightarrow$ use QBF solver?
- PSPACE-h $\rightarrow$ use as modelling language?
- Complexity with quantification?
- More parameters?